

Subdivision into i -packings and S -packing chromatic number of some lattices

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Abstract

An i -packing in a graph G is a set of vertices at pairwise distance greater than i . For a nondecreasing sequence of integers $S = (s_1, s_2, \dots)$, the S -packing chromatic number of a graph G is the least integer k such that there exists a coloring of G into k colors where each set of vertices colored i , $i = 1, \dots, k$, is an s_i -packing. This paper describes various subdivisions of an i -packing into j -packings ($j > i$) for the hexagonal, square and triangular lattices. These results allow us to bound the S -packing chromatic number for these graphs, with more precise bounds and exact values for sequences $S = (s_i, i \in \mathbb{N}^*)$, $s_i = d + \lfloor (i-1)/n \rfloor$.

1 Introduction

Let $G = (V, E)$ be a (finite or infinite) graph and let $N(u) = \{v \in V(G) \mid uv \in E(G)\}$ be the set of neighbors of vertex u . A set $X_i \subseteq V(G)$ is an i -packing if for any distinct pair $u, v \in X_i$, $d(u, v) > i$, where $d(u, v)$ denotes the usual shortest path distance between u and v . We will use X_i to refer to an i -packing in a graph G . A k -coloring c of G is a map from $V(G)$ to $\{1, \dots, k\}$ such that for every pair of adjacent vertices (u, v) , we have $c(u) \neq c(v)$. For a graph G and a k -coloring c of G , let c_i be $\{u \in V(G) \mid c(u) = i\}$. The smallest integer k such that there exists a k -coloring of G for which for every i , with $1 \leq i \leq k$, c_i is a i -packing, is called the *packing chromatic number* of G and is denoted by $\chi_\rho(G)$. This concept was introduced by Goddard *et al.* [7] under the name of broadcast chromatic number. More generally, for a nondecreasing sequence

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of integers $S = (s_1, s_2, \dots)$, an S -packing k -coloring is a k -coloring c of $V(G)$ such that for every i , with $1 \leq i \leq k$, c_i is a s_i -packing. Such a coloring will also simply be called an (s_1, \dots, s_k) -coloring. The S -packing chromatic number of G denoted by $\chi_\rho^S(G)$, is the smallest k such that G admits an S -packing k -coloring. For the sequences $S = (s_i, i \in \mathbb{N}^*)$, with $s_i = d + \lfloor (i-1)/n \rfloor$, we call $\chi_\rho^S(G)$ the (d, n) -packing chromatic number and denote it by $\chi_\rho^{d,n}(G)$. For any connected graph G such that $|V(G)| \geq d+1$, $\chi_\rho^{d,n}(G) \geq d+1$ and $\chi_\rho^{1,1}(G) = \chi_\rho(G)$. For every bipartite graph G , $\chi_\rho^{1,2}(G) = 2$ (a bipartite graph is 2-colorable). Moreover, the smallest n such that $\chi_\rho^{d,n}(G) = n$ corresponds to the d -distant chromatic number [12], *i.e.* the minimum number of d -packings that form a partition of the vertices.

Let P_∞ denote the two-way infinite path, let $\mathbb{Z}^2 = P_\infty \square P_\infty$ denote the planar square lattice (where \square is the Cartesian product), \mathcal{T} denote the planar triangular lattice and \mathcal{H} denote the planar hexagonal lattice. In this article, for an (s_1, s_2, \dots) -coloring of a graph, we prefer to map vertices to the color multiset $\{s_1, s_2, \dots\}$ even if two colors can then be denoted by the same number. This notation allows the reader to directly see to which type of packing the vertex belong depending on its color. When needed, we will denote colors of vertices in different i -packings by i_a, i_b, \dots

1.1 Motivation and related work

Packing colorings in graphs are inspired from frequency planning in wireless systems. The concept of S -packing coloring emphasizes the fact that signals can have different powers similar to the packing coloring but enables the presence of several signals with the same power, providing a more realistic model for the frequency assignment problem.

The packing chromatic number of lattices has been studied by several authors: Soukal and Holub [14] proved that $\chi_\rho(\mathbb{Z}^2) \leq 17$, Ekstein *et al.* [1] that $12 \leq \chi_\rho(\mathbb{Z}^2)$; Fiala *et al.* [4] showed that $\chi_\rho(\mathcal{H}) \leq 7$, $\chi_\rho(\mathbb{Z}^2 \square P_2) = \infty$ and $\chi_\rho(\mathcal{H} \square P_6) = \infty$ and Finbow and Rall [5] proved that $\chi_\rho(\mathcal{T}) = \infty$.

S -packing colorings with sequences S other than $(1, 2, \dots, k)$ first appear in [7, 3]. Goddard and Xu [8] have recently studied S -packing colorings for the infinite path and for square and triangular lattices, determining conditions on the first elements of the sequence for which the graph is or is not S -packing-colorable.

Regarding the complexity, Goddard *et al.* [7] proved that the problem of $(1, 2, 3)$ -packing coloring is polynomial while $(1, 2, 3, 4)$ -packing coloring and $(1, 2, 2)$ -packing coloring are NP-complete. Fiala and Golovach [3] showed that the problem of $(1, 2, \dots, k)$ -coloring is NP-complete for trees. The NP-completeness of $(1, 1, 2)$ -coloring was proved by Goddard and Xu [9] and afterward by Gastineau [6].

While the packing coloring corresponds to an S -packing coloring with a strictly increasing sequence and the d -distant chromatic number corresponds to a constant one, the sequence in the (d, n) -packing coloring also tends to infinity,

$d \backslash n$	1	2	3	4	5	6
1	7 [11, 4]	2	2	2	2	2
2	∞	5 - 8	5	4	4	4
3	∞	15 - 35	9 - 13	8 - 10	7 - 8	6
4	∞	61 - ?	20 - 58	15 - 27	13 - 21	12 - 18
5	∞	∞	37 - ?	25 - ?	21 - ?	19 - ?
8	∞	∞	∞	?	?	?
11	∞	∞	∞	∞	?	?
13	∞	∞	∞	∞	∞	?
16	∞	∞	∞	∞	∞	∞

Table 1: Bounds for (d, n) -packing chromatic numbers of the hexagonal lattice.

but the parameter n allows us to control its growth.

Moreover, one can note that all the S -packing colorings of square and hexagonal lattices published so far have the property that the s_1 -packing is maximum and the other s_i -packings are obtained by subdivisions of s_1 -packings (and are not always maximum). Therefore, we find it interesting to study subdivision of an i -packing into j -packings, $j > i$, in lattices. These subdivisions can in turn be used to describe patterns to obtain an S -packing coloring of a lattice. However, determining the families of graphs G for which for any S such that G is S -colorable, the S -coloring satisfies the above property is satisfied remains an open question. Recently Goddard and Xu [8] proved that there exist nondecreasing sequences S such that P_∞ is S -colorable and in any $(s_1, \dots, \chi_\rho^S(P_\infty))$ -packing coloring of P_∞ , the s_1 -packing is not maximum, showing that for P_∞ , there are sequences S for which the above property is not satisfied.

1.2 Our results

The second section introduces some definitions and results related to density. The third section introduces some subdivision of the lattices into i -packings. The fourth and fifth sections give lower bounds resulting from Section 2 and upper bounds resulting from Section 3 for the S -packing chromatic number and the (d, n) -packing chromatic number of the lattices \mathcal{H} , \mathbb{Z}^2 and \mathcal{T} . Tables 1, 2 and 3 summarize the values obtained in this paper for the (d, n) -packing chromatic number, giving an idea of our results. The emphasized numbers are exact values and all pairs of values are lower and upper bounds. Lower bounds have been calculated from Proposition 2.2 and Propositions 2.5, 2.7 and 2.9. Some of the results for square and triangular lattice have been found independently by Goddard and Xu [8].

$d \backslash n$	1	2	3	4	5	6
1	12 - 17 [1, 14]	2	2	2	2	2
2	∞	11 - 20	7 - 8	6 [8]	5 [8]	5
3	∞	57 - ?	16 - 33	12 - 20	10 - 17	10 - 14
4	∞	∞	44 - ?	25 - 56	20 - 34	18 - 28
5	∞	∞	199 - ?	50 - ?	35 - ?	29 - ?
6	∞	∞	∞	?	?	?
8	∞	∞	∞	∞	?	?
10	∞	∞	∞	∞	∞	?
12	∞	∞	∞	∞	∞	∞

Table 2: Bounds for (d, n) -packing chromatic numbers of the square lattice.

$d \backslash n$	1	2	3	4	5	6
1	∞ [5]	5 - 6 [8]	3	3	3	3
2	∞	127 - ?	14 - ?	10 - 16	9 - 13	8 - 10
3	∞	∞	81 - ?	28 - 72	20 - 38	17 - 26
4	∞	∞	∞	104 - ?	49 - ?	36 - ?
5	∞	∞	∞	∞	?	?
7	∞	∞	∞	∞	∞	?
8	∞	∞	∞	∞	∞	∞

Table 3: Bounds for (d, n) -packing chromatic numbers of the triangular lattice.

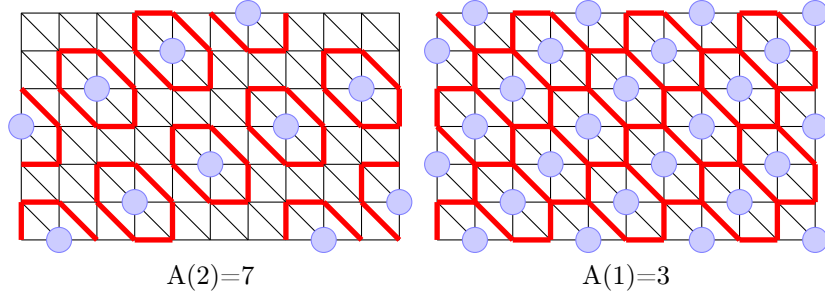


Figure 1: Examples of k -area in \mathcal{T} .

2 Density of i -packings

2.1 Density of an i -packing in a lattice

Let $G = (V, E)$ be a graph, finite or infinite and let n be a positive integer. For a vertex x of G , the ball of radius n centered at x is the set $B_n(x) = \{v \in V(G) | d_G(x, v) \leq n\}$ and the sphere of radius n centered at x is the set $\partial B_n(x) = \{v \in V(G) | d_G(x, v) = n\}$. The density of a set of vertices $X \subseteq V(G)$ is $d(X) = \limsup_{l \rightarrow \infty} \max_{x \in V} \left\{ \frac{|X \cap B_l(x)|}{|B_l(x)|} \right\}$.

The notion of k -area was introduced by Fiala *et al.* [4]. We propose here a slightly modified definition:

Definition 2.1. Let G be a graph, $x \in V(G)$, and let k be a positive integer. The k -area $A(x, k)$ assigned to G is defined by :

$$A(x, k) = \begin{cases} |B_{k/2}(x)| & \text{for } k \text{ even;} \\ |B_{\lfloor k/2 \rfloor}(x)| + \sum_{u \in \partial B_{\lceil k/2 \rceil}(x)} \frac{|N(u) \cap B_{\lfloor k/2 \rfloor}(x)| + |N(u) \cap \partial B_{\lceil k/2 \rceil}(x)|/2}{\deg(u)} & \text{for } k \text{ odd.} \end{cases}$$

For vertex-transitive graphs, the k -areas are the same for all vertices, hence we denote it by $A(k)$.

The motivation for our modification of the notion of k -area with the introduction of the set of neighbors inside the sphere is to have sharper density bounds than the ones obtained by the initial notion of k -area. For the square and the hexagonal lattice the notion coincide as the relation is empty. However, for the triangular lattice, the density bound is smaller: the definition of Fiala *et al.* [4] gives $A(1) = 2$ whereas $A(1) = 3$ in our case since there are adjacent vertices in the sphere (as for every $u \in \partial B_1(x)$, $|N(u) \cap \partial B_1(x)|/2 = 1$, then it adds one to the initial definition of k -area). Figure 1 illustrates this example giving a coverage of the triangular lattice by balls of radius 1. In one case (on the left) the balls are disjoint and in the second case (on the right) each sphere can be shared by several balls. Observe that in the second case, each vertex u in a sphere centered at x has two neighbors, and hence $|N(u) \cap \partial B_1(x)|/2 = 1$.

Proposition 2.2. *Let G be a vertex-transitive graph with finite degree, and i be a positive integer. If X_i is an i -packing in G , then*

$$d(X_i) \leq \frac{1}{A(i)}.$$

Proof. Observe that for arbitrary vertices x and y of an i -packing X_i , the sets $B_{\lfloor i/2 \rfloor}(x)$ and $B_{\lfloor i/2 \rfloor}(y)$ are disjoint, since the vertices x and y are at distance greater than i . Then $d(X_i) < 1/|B_{\lfloor i/2 \rfloor}(x)|$. Assume that i is an odd number and let u be a vertex at distance $\lceil i/2 \rceil$ from x , then u has $\deg(u)$ neighbors. If among these $\deg(u)$ neighbors, k neighbors are in $B_{\lfloor i/2 \rfloor}(x)$ then u can be at distance $\lceil i/2 \rceil$ from only $k/\deg(u)$ vertices in X_i . Hence $d(X_i) < 1/(|B_{\lfloor i/2 \rfloor}(x)| + \sum_{u \in \partial B_{\lceil i/2 \rceil}(x)} \frac{|N(u) \cap B_{\lfloor i/2 \rfloor}(x)|}{\deg(u)})$.

Moreover u and a neighbor v of u in $B_{\lfloor i/2 \rfloor}(x)$ cannot be both at distance $\lceil i/2 \rceil$ from more than 2 vertices in X_i , therefore uv can only belong to two spheres of radius $\lceil i/2 \rceil$ centered at a vertex in X_i . Hence it follows that $d(X_i) < 1/A(i)$. \square

Corollary 2.3. *Let G be a vertex-transitive graph with finite degree, and i be a positive integer. If G has a finite S -packing chromatic number, then*

$$\sum_{i=1}^{\infty} \frac{1}{A(s_i)} \geq 1.$$

Corollary 2.4. *Let G be a vertex-transitive graph with finite degree, and i be a positive integer. If G has a finite (d, n) -packing chromatic number, then*

$$\sum_{i=d}^{\infty} \frac{n}{A(i)} \geq 1.$$

An i -packing X_i is called a *maximized i -packing* if for any other i -packing X'_i , $d(X_i) \geq d(X'_i)$.

2.2 Density of an i -packing in the hexagonal lattice

Proposition 2.5. *Let \mathcal{H} be the hexagonal lattice, x be a vertex in $V(\mathcal{H})$ and n be a positive integer. Then*

1. $|\partial B_n(x)| = 3n$;
2. $|B_n(x)| = \frac{3}{2}n^2 + \frac{3}{2}n + 1$.

Proof. 1. As the set $\partial B_n(x)$ always contains three more vertices than $\partial B_{n-1}(x)$, then $|\partial B_n(x)| = 3n$.

2. The graph \mathcal{H} is 3-regular and so $|B_1(x)| = 4$. Suppose the statement is true for n , then $|B_{n+1}(x)| = |B_n(x)| + |\partial B_{n+1}(x)| = \frac{3}{2}n^2 + \frac{3}{2}n + 1 + 3(n+1) = (\frac{3}{2}n^2 + \frac{3}{2}n + 1) + (\frac{3}{2}n + \frac{3}{2}) + 1 = \frac{3}{2}(n+1)^2 + \frac{3}{2}(n+1) + 1$ and the result follows by induction. \square

Proposition 2.6. *Let \mathcal{H} be the hexagonal lattice and k be a positive integer. Then*

1. $A(2k) = \frac{3}{2}k^2 + \frac{3}{2}k + 1$;
2. $A(4k + 1) = 6k^2 + 6k + 2$;
3. $A(4k + 3) = 6k^2 + 12k + 6$.

Proof. 1. The first property results easily from Proposition 2.5.

2. If $n = 4k + 1$, then Proposition 2.5 gives $|B_{2k}(x)| = \frac{3}{2}(2k)^2 + \frac{3}{2}(2k) + 1 = 6k^2 + 3k + 1$. For every vertex y in $\partial B_{2k+1}(x)$, y has no neighbor in $\partial B_{2k+1}(x)$ other than itself, so $|N(y) \cap \partial B_{2k+1}(x)| = 0$. We have to distinguish two kinds of vertices: $3k$ vertices have two neighbors in $B_{2k}(x)$ and $|\partial B_{2k+1}(x)| - 3k = 3k + 3$ vertices have one neighbor in $B_{2k}(x)$. Therefore, $|A(4k + 1)| = 6k^2 + 3k + 1 + \frac{6k}{3} + \frac{3k+3}{3} = 6k^2 + 6k + 2$.

3. If $n = 4k + 3$, then Proposition 2.5 gives $|B_{2k+1}(x)| = \frac{3}{2}(2k + 1)^2 + \frac{3}{2}(2k + 1) + 1 = 6k^2 + 9k + 4$. For every vertex y in $\partial B_{2k+2}(x)$, y has no neighbor in $\partial B_{2k+2}(x)$ other than itself, so $|N(y) \cap \partial B_{2k+2}(x)| = 0$. We have to distinguish two kinds of vertices: $3k$ vertices have two neighbors in $B_{2k+2}(x)$ and $|\partial B_{2k+2}(x)| - 3k = 3k + 6$ vertices have one neighbor in $B_{2k}(x)$. Hence, we have $|A(4k + 3)| = 6k^2 + 9k + 4 + \frac{6k}{3} + \frac{3k+6}{3} = 6k^2 + 12k + 6$. \square

Note that this result appeared in the article of Goddard and Xu [8].

2.3 Density of an i -packing in the square lattice

Proposition 2.7. *Let \mathbb{Z}^2 be the square lattice, x be a vertex in $V(\mathbb{Z}^2)$ and n be a positive integer. Then*

1. $|\partial B_n(x)| = 4n$;
2. $|B_n(x)| = 2n^2 + 2n + 1$.

Proof. 1. As the set $\partial B_n(x)$ always contains four more vertices than $\partial B_{n-1}(x)$, then $|\partial B_n(x)| = 4n$.

2. The graph \mathbb{Z}^2 is 4-regular and so $|B_1(x)| = 5$. Suppose the statement is true for n , then $|B_{n+1}(x)| = |B_n(x)| + |\partial B_{n+1}(x)| = 2n^2 + 2n + 1 + 4n + 4 = 2(n + 1)^2 + 6n + 5 - 4n - 2 = 2(n + 1)^2 + 2(n + 1) + 1$ and the result follows by induction. \square

Proposition 2.8. *Let \mathbb{Z}^2 be the square lattice and k be a positive integer. Then*

1. $A(2k) = 2k^2 + 2k + 1$;
2. $A(2k + 1) = 2k^2 + 4k + 2$.

Proof. 1. The first property results easily from Proposition 2.7.

2. If $n = 2k + 1$, then Proposition 2.7 gives $|B_k(x)| = k^2 + 2k + 1$. For every vertex y in $\partial B_{k+1}(x)$, y has no neighbor in $\partial B_{k+1}(x)$ other than itself, so

$|N(y) \cap \partial B_{k+1}(x)| = 0$. We have to distinguish two kinds of vertices: $4k$ vertices have two neighbors in $B_k(x)$ and 4 vertices have one neighbor in $B_k(x)$. Hence, we have $|A(2k+1)| = 2k^2 + 2k + 1 + 2\frac{4k}{4} + \frac{4}{4} = 2k^2 + 4k + 2$. \square

Note that this result appeared implicitly in the article of Fiala et al. [4].

2.4 Density of an i -packing in the triangular lattice

Proposition 2.9. *Let \mathcal{T} be the triangular lattice, x be a vertex in $V(\mathcal{T})$ and n be a positive integer. Then*

1. $|\partial B_n(x)| = 6n$;
2. $|B_n(x)| = 3n^2 + 3n + 1$.

Proof. 1. As the set $\partial B_n(x)$ always contains six more vertices than $\partial B_{n-1}(x)$, then $|\partial B_n(x)| = 6n$.

2. The graph \mathcal{T} is 6-regular and so $|B_1(x)| = 7$. Suppose the statement is true for n , then $|B_{n+1}(x)| = |B_n(x)| + |\partial B_{n+1}(x)| = 3n^2 + 3n + 1 + 6n + 6 = 3(n^2 + 1) + 3n + 1 + 6 - 3 = 3(n^2 + 1) + 3(n + 1) + 1$ and the result follows by induction. \square

Proposition 2.10. *Let \mathcal{T} be the triangular lattice and k be a positive integer. Then*

1. $A(2k) = 3k^2 + 3k + 1$;
2. $A(2k+1) = 3k^2 + 6k + 3$.

Proof. 1. The first property result easily from Proposition 2.9.

2. If $n = 2k + 1$, then Proposition 2.9 gives $|B_k(x)| = 3k^2 + 3k + 1$. For every vertex y in $\partial B_{k+1}(x)$, y has two neighbors in $\partial B_{k+1}(x)$ other than itself, so $|N(y) \cap \partial B_{2k+1}(x)| = 2$. We have to distinguish two kinds of vertices: $6k$ vertices have two neighbors in $B_k(x)$ and six vertices have one neighbor in $B_k(x)$. Hence, we have $|A(2k+1)| = 3k^2 + 3k + 1 + \frac{6k+6}{6} + 2\frac{6k}{6} + \frac{6}{6} = 3k^2 + 6k + 3$. \square

3 Subdivision of an i -packing in \mathcal{H} , \mathbb{Z}^2 and \mathcal{T}

3.1 Subdivision of a 2-packing in \mathcal{H}

Let X_2 be the (unique) maximized 2-packing in \mathcal{H} represented in Figure 2. Note that $d(X_2) = 1/A(2) = 1/4$ and remark that four 2-packings form a partition of \mathcal{H} if we translate X_2 three times.

The hexagonal lattice can be seen as a subgraph of the square lattice. In fact in Figure 2, \mathcal{H} is represented as subgraph of the usual representation of the square lattice. In the square lattice, we can choose one vertex as the origin and all the other vertices can be nominated by a Cartesian coordinate. In every description of \mathcal{H} , our origin $(0, 0)$ will be a vertex in the packing that we want to

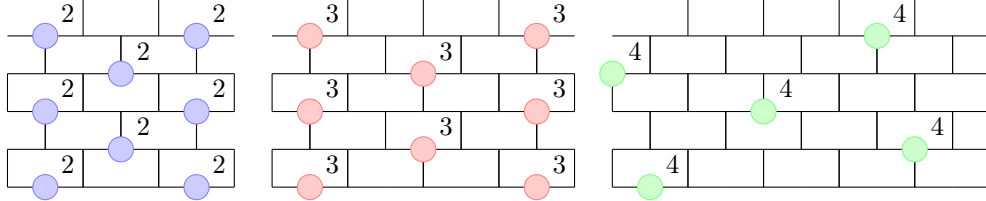


Figure 2: The sets X_2 (2-packing), X_3 (3-packing) and X_4 (4-packing) in \mathcal{H} .

describe such that there is no edge between $(0, 0)$ and $(0, 1)$. In fact we illustrate packings with a figure in this subsection but it will not be the case after; we will use Cartesian coordinates in order to describe a packing. For example, X_2 from Figure 2 is the set of vertices: $X_2 = \{(2x + 4y, x) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$.

In Appendix A, we recall a proposition about distance in the hexagonal lattice from Jacko and Jendrol [10]. This proposition is useful to verify that a set is an i -packing. These verifications are left to the reader in the remaining propositions.

Proposition 3.1. *Let $k > 0$ and $m > 0$ be integers. There exist:*

1. k^2 $(3k - 1)$ -packings that form a partition of X_2 ;
2. $2k^2$ $(4k - 1)$ -packings that form a partition of X_2 ;
3. two $(3 \times 2k - 1)$ -packings that form a partition of a $(4k - 1)$ -packing from Point 2;
4. m^2 $(3mk - 1)$ -packings that form a partition of a $(3k - 1)$ -packing from Point 1;
5. m^2 $(4mk - 1)$ -packings that form a partition of a $(4k - 1)$ -packing from Point 2.

Proof. 1. Let A_k be the $(3k - 1)$ -packing defined by $A_k = \{(2kx + 4ky, kx) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$. Let $\mathcal{F} = \{(2i + 4j, i) \mid i, j \in \{0, \dots, k - 1\}\}$ be a family of k^2 vectors. Make k^2 copies of the set A_k and translate each one by a vector from \mathcal{F} to obtain a partition of X_2 .

2. Let B_k be the $(4k - 1)$ -packing defined by $B_k = \{(4kx, 2ky) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$. Let $\mathcal{F} = \{(4i + 2a, 2j + a) \mid i, j \in \{0, \dots, k - 1\}, a \in \{0, 1\}\}$ be a family of $2k^2$ vectors. Make $2k^2$ copies of the set B_k and translate each one by a vector from \mathcal{F} to obtain a partition of X_2 .

3. Note that $A_{2k} \subseteq B_k$ and if A'_{2k} is A_{2k} translated by the vector $(0, 2k)$, then $A'_{2k} \cup A_{2k} = B_k$.

4. Note that $A_{mk} \subseteq A_k$. Let $\mathcal{F} = \{(2mki + 4mkj, mki) \mid i, j \in \{0, \dots, m - 1\}\}$ be a family of m^2 vectors. Make m^2 copies of the set A_{mk} and translate each one by a vector from \mathcal{F} to obtain a partition of A_k .

5. Note that $B_{mk} \subseteq B_k$. Let $\mathcal{F} = \{(4mki, 2mkj) \mid i, j \in \{0, \dots, m - 1\}\}$ be a

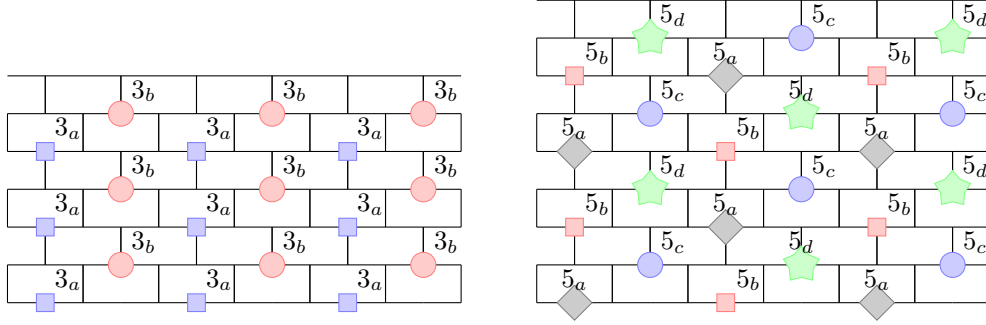


Figure 3: Two 3-packings forming a partition of X_2 (on the left) and four 5-packings forming a partition of X_2 (on the right).

family of m^2 vectors. Make m^2 copies of the set B_{mk} and translate each one by a vector from \mathcal{F} to obtain a partition of B_k . \square

Figure 3 illustrates a partition of X_2 from Points 1 and 2 for $k = 1$. In the remaining of the section, the proofs of decomposition of a set X will be resumed in a table and the proofs of properties similar from those from Points 3, 4 and 5 will be left to the reader.

3.2 Subdivision of a 3-packing in \mathcal{H}

Let $X_3 = \{(3x + 6y, x) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the maximized 3-packing in \mathcal{H} from Figure 2. Note that $d(X_3) = 1/A(3) = 1/6$ and that six 3-packings form a partition of \mathcal{H} if we translate X_3 five times.

Proposition 3.2. *Let $k > 0$ and $m > 0$ be integers. There exist:*

1. k^2 $p_{1,k}$ -packings, $p_{1,k} = (4k - 1)$, that form a partition of X_3 ;
2. $3k^2$ $p_{2,k}$ -packings, $p_{2,k} = (6k - 1)$, that form a partition of X_3 ;
3. $8k^2$ $p_{3,k}$ -packings, $p_{3,k} = (10k - 1)$, that form a partition of X_3 ;
4. $24k^2$ $p_{4,k}$ -packings, $p_{4,k} = (18k - 1)$, that form a partition of X_3 ;
5. m^2 $p_{j,mk}$ -packings that form a partition of a $p_{j,k}$ -packing from Point j , for $j \in \{1, \dots, 4\}$;
6. three $(4 \times 3k - 1)$ -packings that form a partition of a $(6k - 1)$ -packing from Point 2;
7. two $(4 \times 4k - 1)$ -packings that form a partition of a $(10k - 1)$ -packing from Point 3;
8. four 17-packings and six 23-packings that form a partition of every 5-packing from Point 2.

Proof. The proof is resumed in Table B.4, this table contains: in which i -packing X will be decomposed (Column 1), the number of i -packings needed to form a partition of X (Column 2), the description of an i -packing with Cartesian coordinates (assuming x and y are integers) (Column 3) and the family of translation vectors (Column 4). We assume that if we do copies of this i -packing and we translate each one by one of these vectors. Afterward, we obtain a partition of X in i -packings. \square

3.3 Subdivision of a 4-packing in \mathcal{H}

Let $X_4 = \{(3x + 7y, 2x + y) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the 4-packing in \mathcal{H} from Figure 2. Note that $d(X_4) = 1/11$ and that $1/A(4) = 1/10$. However, we claim that a 4-packing with density $1/10$ does not exist. Note that eleven 4-packings form a partition of \mathcal{H} if we translate X_4 ten times.

Proposition 3.3. *Let $k > 0$ and $m > 0$ be integers. There exist:*

1. k^2 $p_{1,k}$ -packings, $p_{1,k} = (5k - 1)$, that form a partition of X_4 ;
2. $2k^2$ $p_{2,k}$ -packings, $p_{2,k} = (6k - 1)$, that form a partition of X_4 ;
3. $3k^2$ $p_{3,k}$ -packings, $p_{3,k} = (8k - 1)$, that form a partition of X_4 ;
4. $6k^2$ $p_{4,k}$ -packings, $p_{4,k} = (11k - 1)$, that form a partition of X_4 ;
5. m^2 $p_{j,mk}$ -packings that form a partition of a $p_{j,k}$ -packing from Point j , for $j \in \{1, \dots, 4\}$;
6. two $(5 \times 2k - 1)$ -packings that form a partition of a $(6k - 1)$ -packing from Point 2;
7. two $(6k - 1)$ -packings that form a partition of a $(5k - 1)$ -packing from Point 1;
8. three $(5 \times 3k - 1)$ -packings that form a partition of a $(8k - 1)$ -packing from Point 2;
9. three $(8k - 1)$ -packings that form a partition of a $(5k - 1)$ -packing from Point 1.

Proof. See Table B.5. \square

3.4 Subdivision of a 2-packing in \mathbb{Z}^2

In the square lattice, we can choose one vertex as the origin and all the other vertices will be nominated by Cartesian coordinates. In all our representations our origin $(0, 0)$ will be in the packing that we want to describe. Let $X_2 = \{(2x + y, x + 3y) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the maximized 2-packing in \mathbb{Z}^2 from Figure 4. Note that $d(X_2) = 1/A(2) = 1/5$ and that five 2-packings form a partition of \mathbb{Z}^2 if we translate X_2 four times.

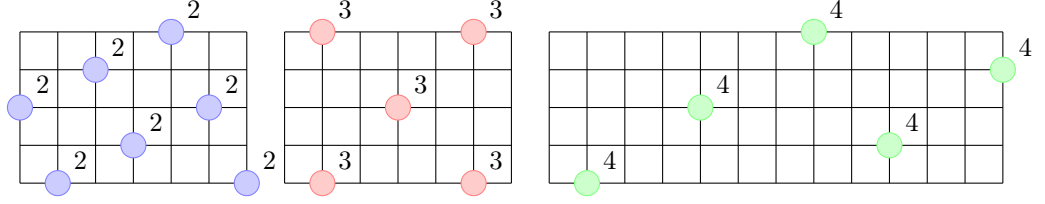


Figure 4: The sets X_2 (2-packing), X_3 (3-packing) and X_4 (4-packing) in \mathbb{Z}^2 .

Proposition 3.4. *Let $k > 0$, $m > 0$ be integers. There exist:*

1. k^2 $(3k - 1)$ -packings that form a partition of X_2 ;
2. $2k^2$ $(4k - 1)$ -packings that form a partition of X_2 ;
3. two $(3 \times 2k - 1)$ -packings that form a partition of a $(4k - 1)$ -packing from Point 2;
4. two $(4k - 1)$ -packings that form a partition of a $(3k - 1)$ -packing from Point 1;
5. m^2 $(3mk - 1)$ -packings that form a partition of a $(3k - 1)$ -packing from Point 1;
6. m^2 $(4mk - 1)$ -packings that form a partition of a $(4k - 1)$ -packing from Point 2.

Proof. See Table B.6. □

3.5 Subdivision of a 3-packing in \mathbb{Z}^2

Let $X_3 = \{(2x + 4y, 2y) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the maximized 3-packing in \mathbb{Z}^2 from Figure 4. Note that $d(X_3) = 1/A(3) = 1/8$ and that eight 3-packings form a partition of \mathbb{Z}^2 if we translate X_3 seven times.

Proposition 3.5. *Let $k > 0$ and $m > 0$ be integers. There exist:*

1. k^2 $(4k - 1)$ -packings that form a partition of X_3 ;
2. m^2 $(4mk - 1)$ -packings that form a partition of a $(4k - 1)$ -packing from Point 1.

Proof. See Table B.6. □

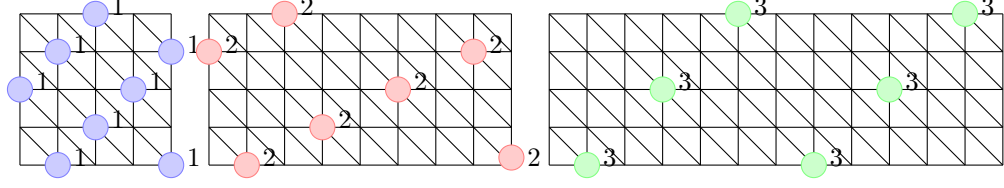


Figure 5: The sets X_1 (1-packing), X_2 (2-packing) and X_3 (3-packing) in \mathcal{T} .

3.6 Subdivision of a 4-packing in \mathbb{Z}^2

Let $X_4 = \{(3x + 8y, 2x + y) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the maximized 4-packing in \mathbb{Z}^2 from Figure 4. Note that $d(X_4) = 1/A(4) = 1/13$ and that thirteen 4-packings form a partition of \mathbb{Z}^2 if we translate X_4 twelve times.

Proposition 3.6. *Let $k > 0$, $m > 0$ be integers. There exist:*

1. $k^2 (5k - 1)$ -packings that form a partition of X_4 ;
2. $2k^2 (6k - 1)$ -packings that form a partition of X_4 ;
3. two $(5 \times 2k - 1)$ -packings that form a partition of a $(6k - 1)$ -packing from Point 2;
4. two $(6k - 1)$ -packings that form a partition of a $(5k - 1)$ -packing from Point 1;
5. $m^2 (5mk - 1)$ -packings that form a partition of a $(5k - 1)$ -packing from Point 1;
6. $m^2 (6mk - 1)$ -packings that form a partition of a $(6k - 1)$ -packing from Point 2.

Proof. See Table B.6. □

3.7 Subdivision of an independent set in \mathcal{T}

The square lattice can be seen as a subgraph of the triangular lattice. In fact in Figure 5, the triangular lattice is represented as a supergraph of the square lattice. Therefore, we can choose one vertex as the origin and all the other vertices will be nominated by Cartesian coordinates. In all our representations our origin $(0, 0)$ will be a vertex such that $(0, 0)$ has $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$, $(-1, 1)$ and $(1, -1)$ as neighbors. Let $X_1 = \{(x + 3y, x) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the (unique) maximized independent set (1-packing) in \mathcal{T} from Figure 5. Note that $d(X_1) = 1/A(1) = 1/3$ and that three independent sets form a partition of \mathcal{T} if we translate X_1 two times.

Proposition 3.7. *Let $k > 0$ and $m > 0$ be integers. There exist:*

1. k^2 $(2k - 1)$ -packings that form a partition of X_1 ;
2. $3k^2$ $(3k - 1)$ -packings that form a partition of X_1 ;
3. three $(3k - 1)$ -packings that form a partition of a $(2k - 1)$ -packing from Point 1;
4. m^2 $(2mk - 1)$ -packings that form a partition of a $(2k - 1)$ -packing from Point 1;
5. m^2 $(3mk - 1)$ -packings that form a partition of a $(3k - 1)$ -packing from Point 2.

Proof. See Table B.7. □

3.8 Subdivision of a 2-packing in \mathcal{T}

Let $X_2 = \{(2x - y, x + 3y) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the maximized 2-packing in \mathcal{T} from Figure 5. Note that $d(X_2) = 1/A(2) = 1/7$ and that seven 2-packings form a partition of \mathcal{T} if we translate X_2 six times.

Proposition 3.8. *Let $k > 0$ and $m > 0$ be integers. There exist:*

1. k^2 $(3k - 1)$ -packings that form a partition of X_2 ;
2. m^2 $(3mk - 1)$ -packings that form a partition of a $(3k - 1)$ -packing from Point 1.

Proof. See Table B.7. □

3.9 Subdivision of a 3-packing in \mathcal{T}

Let $X_3 = \{(2x + 6y, 2x) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$ be the maximized 3-packing in \mathcal{T} from Figure 5. Note that $d(X_3) = 1/A(3) = 1/12$ and that twelve 3-packings form a partition of \mathcal{T} if we translate X_3 eleven times.

Proposition 3.9. *Let $k > 0$ and $m > 0$ be integers. There exist:*

1. k^2 $(4k - 1)$ -packings that form a partition of X_3 ;
2. $3k^2$ $(6k - 1)$ -packings that form a partition of X_3 ;
3. three $(4 \times 3k - 1)$ -packings that form a partition of a $(6k - 1)$ -packing from Point 1;
4. m^2 $(4mk - 1)$ -packings that form a partition of a $(4k - 1)$ -packing from Point 1;
5. m^2 $(6mk - 1)$ -packings that form a partition of a $(6k - 1)$ -packing from Point 2.

Proof. See Table B.7. □

4 S -packing chromatic number

4.1 General properties

In the previous section, we obtained several properties of subdivision of an i -packings in a lattice. This section illustrates general properties obtained on the S -packing chromatic number using only a small part of these properties. For a given sequence S , one can find other colorings of a lattice using properties from the previous section.

Corollary 4.1. *Let $a_0 = 1$. If $s_1 = 2$ and there exist three integers $1 < a_1 < \dots < a_3$ and three integers k_1, \dots, k_3 such that $s_{a_i} \leq 3k_i - 1$ and $a_i - a_{i-1} \geq k_i^2$ or $s_{a_i} \leq 4k_i - 1$ and $a_i - a_{i-1} \geq 2k_i^2$ for $i \in \{1, \dots, 3\}$ then $\chi_\rho^S(\mathcal{H}) \leq a_3$.*

This corollary can be useful to find upper bounds for a given sequence. For example, if $S = (2, 2, 2, 2, \dots)$, then taking $a_1 = 2$, $a_2 = 3$ and $a_3 = 4$, Corollary 4.1 gives us $\chi_\rho^S(\mathcal{H}) \leq 4$ (this result is in fact treated in next subsection). Similarly, for the sequence $S = (2, 3, 3, 5, 5, 5, 5, 7, 7, 7, 7, 7, 7, \dots)$, then taking $a_1 = 3$, $a_2 = 7$ and $a_3 = 15$, Corollary 4.1 gives us $\chi_\rho^S(\mathcal{H}) \leq 15$. There are similar results for $s_1 = 3$ or $s_1 = 4$ using Propositions 3.3 and 3.4.

For the two remaining lattices, the two following properties are given for \mathbb{Z}^2 with $s_1 = 2$ and for \mathcal{T} with $s_1 = 1$. There exist similar properties for \mathbb{Z}^2 with $s_1 = 3$ or 4 using Propositions 3.5 and 3.6 and for \mathcal{T} with $s_1 = 2$ or 3 using Propositions 3.8 and 3.9.

Corollary 4.2. *Let $a_0 = 1$. If $s_1 = 2$ and there exist four integers $1 < a_1 < \dots < a_4$ and four integers k_1, \dots, k_4 such that $s_{a_i} \leq 3k_i - 1$ and $a_i - a_{i-1} \geq k_i^2$ or $s_{a_i} \leq 4k_i - 1$ and $a_i - a_{i-1} \geq 2k_i^2$ for $i \in \{1, \dots, 4\}$ then $\chi_\rho^S(\mathbb{Z}^2) \leq a_4$.*

Corollary 4.3. *Let $a_0 = 1$. If $s_1 = 1$ and there exist two integers $1 < a_1 < a_2$ and two integers k_1 and k_2 such that $s_{a_i} \leq 2k_i - 1$ and $a_i - a_{i-1} \geq k_i^2$ or $s_{a_i} \leq 3k_i - 1$ and $a_i - a_{i-1} \geq 3k_i^2$ for $i \in \{1, \dots, 2\}$ then $\chi_\rho^S(\mathcal{T}) \leq a_2$.*

4.2 S -packing chromatic number and distance coloring

Jacko and Jendrol [10], Fertin *et al.* [2] and Ševčíková [13] have studied distance colorings of \mathcal{H} , \mathbb{Z}^2 and \mathcal{T} respectively. The following propositions comes from their work and can be translated in S -packing coloring:

Proposition 4.4 ([10]). *Let n and d be integers. The minimum n such that $s_1 = d$, $s_n = d$ and $\chi_\rho^S(\mathcal{H}) = n$ is given by*

$$n = \begin{cases} \lceil \frac{3}{8}(d+1)^2 \rceil & \text{for } d \text{ odd;} \\ \lceil \frac{3}{8}(d+4/3)^2 \rceil & \text{for } d \text{ even.} \end{cases}$$

Proposition 4.5 ([2]). *Let n and d be integers. The minimum n such that $s_1 = d$, $s_n = d$ and $\chi_\rho^S(\mathbb{Z}^2) = n$ is given by*

$$n = \begin{cases} \frac{1}{2}(d+1)^2 & \text{for } d \text{ odd;} \\ \frac{1}{2}((d+1)^2 + 1) & \text{for } d \text{ even.} \end{cases}$$

Proposition 4.6 ([13]). *Let n and d be integers. The minimum n such that $s_1 = d$, $s_n = d$ and $\chi_\rho^S(\mathcal{T}) = n$ is given by*

$$n = \left\lceil \frac{3}{4}(d+1)^2 \right\rceil.$$

5 (d, n) -packing chromatic number

5.1 Hexagonal lattice

Proposition 5.1. $\chi_\rho^{2,1}(\mathcal{H}) = \infty$, $\chi_\rho^{5,2}(\mathcal{H}) = \infty$, $\chi_\rho^{8,3}(\mathcal{H}) = \infty$, $\chi_\rho^{11,4}(\mathcal{H}) = \infty$, $\chi_\rho^{13,5}(\mathcal{H}) = \infty$ and $\chi_\rho^{16,6}(\mathcal{H}) = \infty$.

Proof. Let \mathcal{H} be the hexagonal lattice and k be an integer, $k \geq 16$.

$$\sum_{i=1}^k \frac{1}{A(i)} = \sum_{i=1}^n \frac{1}{A(2i)} + \sum_{i=0}^n \frac{1}{A(4i+1)} + \sum_{i=0}^k \frac{1}{A(4i+3)} = \sum_{i=1}^k \frac{1}{\frac{3}{2}i^2 + \frac{3}{2}i+1} + \sum_{i=0}^k \frac{1}{6i^2+6i+2} + \sum_{i=0}^k \frac{1}{6i^2+12i+6} < \frac{2}{15}\sqrt{15}\pi \tanh(\frac{1}{6}\pi\sqrt{15}) + \frac{1}{6}\sqrt{3}\pi \tanh(\frac{1}{6}\pi\sqrt{3}) + \frac{1}{36}\pi^2 - 1 < 1.494.$$

Therefore: $\sum_{i=2}^k \frac{1}{A(i)} < 1.494 - \frac{1}{A(1)} < 0.994 < 1$, $\sum_{i=5}^k \frac{2}{A(i)} < 2(1.494 - \sum_{i=1}^4 \frac{1}{A(i)}) < 0.955 < 1$, $\sum_{i=8}^k \frac{3}{A(i)} < 3(1.494 - \sum_{i=1}^7 \frac{1}{A(i)}) < 0.935 < 1$, $\sum_{i=11}^k \frac{4}{A(i)} < 4(1.494 - \sum_{i=1}^{10} \frac{1}{A(i)}) < 0.925 < 1$ and $\sum_{i=13}^k \frac{5}{A(i)} < 5(1.494 - \sum_{i=1}^{12} \frac{1}{A(i)}) < 0.986 < 1$, $\sum_{i=16}^k \frac{6}{A(i)} < 6(1.494 - \sum_{i=1}^{15} \frac{1}{A(i)}) < 0.968 < 1$. Corollary 2.3 allows us to conclude. \square

Proposition 5.2. $\chi_\rho^{2,2}(\mathcal{H}) \leq 8$, $\chi_\rho^{2,3}(\mathcal{H}) \leq 5$ and $\forall n \geq 4$, $\chi_\rho^{2,n}(\mathcal{H}) = 4$.

Proof. Using Proposition 3.1, we define a $(2, n)$ -packing coloring of \mathcal{H} for each $n = 2, 3$ and $n \geq 4$. \mathcal{H} can be partitioned into four 2-packings, the first two ones can be colored by color 2, the third one by two colors 3 and the last one by four colors under 5, it will be two 4 and two 5, to conclude $\chi_\rho^{2,2}(\mathcal{H}) \leq 8$. \mathcal{H} can be partitioned into four 2-packings, the first three ones can be colored by colors 2 and the third one by two colors 3, to conclude $\chi_\rho^{2,3}(\mathcal{H}) \leq 5$. \mathcal{H} can be partitioned into four 2-packings, hence $\forall n \geq 4$, $\chi_\rho^{2,n}(\mathcal{H}) = 4$. \square

The following table summarizes the colorings defined in the above proof. The symbol P in the table refers to the packings we use and how we subdivide them into i -packings (A_i is an i -packing) and the symbol C refers to the associated colors we use for each i -packing. By $k \times A_i$ we mean we use k i -packings, and by $k \times i$ we mean we use k colors i . In the rest of the paper, similar proofs will be only described by a table using the same format than this one.

(2,2)-packing	P	$2 \times X_2$	X_2	X_2
	C	2×2	$2 \times A_3$	$4 \times A_5$
(2,3)-packing	P	$3 \times X_2$	X_2	
	C	3×2	$2 \times A_3$	

Proposition 5.3. $\chi_\rho^{3,2}(\mathcal{H}) \leq 35$, $\chi_\rho^{3,3}(\mathcal{H}) \leq 13$, $\chi_\rho^{3,4}(\mathcal{H}) \leq 10$, $\chi_\rho^{3,5}(\mathcal{H}) \leq 8$ and $\forall n \geq 6$, $\chi_\rho^{3,n}(\mathcal{H}) = 6$.

Proof. Using Proposition 3.2, we define a $(3, n)$ -packing coloring of \mathcal{H} for each $n = 2, 3, 4, 5$ and $n \geq 6$. \mathcal{H} can be partitioned into six 3-packings, hence $\forall n \geq 6$, $\chi_\rho^{3,n}(\mathcal{H}) = 6$. The other colorings are described in Table C.8. \square

Proposition 5.4. $\chi_\rho^{4,3}(\mathcal{H}) \leq 58$, $\chi_\rho^{4,4}(\mathcal{H}) \leq 27$, $\chi_\rho^{4,5}(\mathcal{H}) \leq 21$, $\chi_\rho^{4,6}(\mathcal{H}) \leq 18$ and from [10] $\forall n \geq 11$, $\chi_\rho^{4,n}(\mathcal{H}) = 11$.

Proof. Using Proposition 3.3, we define a $(4, n)$ -packing coloring of \mathcal{H} for each $n = 3, 4, 5, 6$ and $n \geq 11$. \mathcal{H} can be partitioned into eleven 4-packings. The other colorings are described in Table C.9. \square

5.2 Square lattice

Proposition 5.5. $\chi_\rho^{2,1}(\mathbb{Z}^2) = \infty$, $\chi_\rho^{4,2}(\mathbb{Z}^2) = \infty$, $\chi_\rho^{6,3}(\mathbb{Z}^2) = \infty$, $\chi_\rho^{8,4}(\mathbb{Z}^2) = \infty$, $\chi_\rho^{10,5}(\mathbb{Z}^2) = \infty$ and $\chi_\rho^{12,6}(\mathbb{Z}^2) = \infty$.

Proof. Let \mathbb{Z}^2 be the square lattice and k be an integer, $k \geq 12$.

$$\sum_{i=1}^k \frac{1}{A(i)} = \sum_{i=1}^k \frac{1}{A(2i)} + \sum_{i=0}^k \frac{1}{A(2i+1)} = \sum_{i=1}^k \frac{1}{2i^2+2i+1} + \sum_{i=0}^k \frac{1}{2i^2+4i+2} < \frac{1}{2}\pi \tanh(\frac{1}{2}\pi) + \frac{1}{12}\pi^2 - 1 < 1.264.$$

Therefore: $\sum_{i=2}^k \frac{1}{A(i)} < 1.264 - \frac{1}{A(1)} < 0.764 < 1$, $\sum_{i=4}^k \frac{2}{A(i)} < 2(1.264 - \sum_{i=1}^3 \frac{1}{A(i)}) < 0.877 < 1$, $\sum_{i=6}^k \frac{3}{A(i)} < 3(1.264 - \sum_{i=1}^5 \frac{1}{A(i)}) < 0.917 < 1$, $\sum_{i=8}^k \frac{4}{A(i)} < 4(1.264 - \sum_{i=1}^7 \frac{1}{A(i)}) < 0.938 < 1$, $\sum_{i=10}^k \frac{5}{A(i)} < 5(1.264 - \sum_{i=1}^9 \frac{1}{A(i)}) < 0.951 < 1$ and $\sum_{i=12}^k \frac{6}{A(i)} < 6(1.264 - \sum_{i=1}^{11} \frac{1}{A(i)}) < 0.959 < 1$. Corollary 2.3 allows us to conclude. \square

Proposition 5.6. $\chi_\rho^{2,2}(\mathbb{Z}^2) \leq 20$, $\chi_\rho^{2,3}(\mathbb{Z}^2) \leq 8$, $\chi_\rho^{2,4}(\mathbb{Z}^2) \leq 6$ and $\forall n \geq 5$, $\chi_\rho^{2,n}(\mathbb{Z}^2) = 5$.

Proof. Using Proposition 3.4, we define a $(2, n)$ -packing coloring of \mathbb{Z}^2 for each $n = 2, 3, 4$ and $n \geq 5$. \mathbb{Z}^2 can be partitioned into five 2-packings, hence $\forall n \geq 5$, $\chi_\rho^{2,n}(\mathbb{Z}^2) = 5$. The other colorings are described in Table C.10. \square

Soukal and Holub [14] have proven that $\chi_\rho^{1,1}(\mathbb{Z}^2) \leq 17$, and proposed a 24×24 pattern in order to color the square lattice. Their pattern is recalled in Figure 6.

1	2	1	3	1	2	1	10	1	4	1	9	1	2	1	3	1	2	1	5	1	4	1	14
7	1	5	1	6	1	3	1	2	1	3	1	8	1	5	1	4	1	3	1	2	1	3	1
1	3	1	2	1	4	1	7	1	5	1	2	1	3	1	2	1	11	1	6	1	10	1	2
4	1	9	1	3	1	2	1	3	1	6	1	4	1	7	1	3	1	2	1	3	1	5	1
1	2	1	15	1	5	1	11	1	2	1	3	1	2	1	17	1	5	1	4	1	2	1	3
6	1	3	1	2	1	3	1	4	1	14	1	5	1	3	1	2	1	3	1	7	1	8	1
1	5	1	4	1	16	1	2	1	3	1	2	1	10	1	4	1	13	1	2	1	3	1	2
3	1	2	1	3	1	6	1	5	1	7	1	3	1	2	1	3	1	9	1	5	1	4	1
1	7	1	10	1	2	1	3	1	2	1	4	1	6	1	5	1	2	1	3	1	2	1	11
2	1	3	1	5	1	4	1	8	1	3	1	2	1	3	1	7	1	4	1	6	1	3	1
1	4	1	2	1	3	1	2	1	9	1	5	1	11	1	2	1	3	1	2	1	12	1	5
3	1	6	1	13	1	7	1	3	1	2	1	3	1	4	1	8	1	5	1	3	1	2	1
1	2	1	3	1	2	1	5	1	4	1	15	1	2	1	3	1	2	1	10	1	4	1	9
8	1	5	1	4	1	3	1	2	1	3	1	7	1	5	1	6	1	3	1	2	1	3	1
1	3	1	2	1	11	1	6	1	10	1	2	1	3	1	2	1	4	1	7	1	5	1	2
4	1	7	1	3	1	2	1	3	1	5	1	4	1	9	1	3	1	2	1	3	1	6	1
1	2	1	17	1	5	1	4	1	2	1	3	1	2	1	14	1	5	1	11	1	2	1	3
5	1	3	1	2	1	3	1	7	1	8	1	6	1	3	1	2	1	3	1	4	1	15	1
1	10	1	4	1	9	1	2	1	3	1	2	1	5	1	4	1	16	1	2	1	3	1	2
3	1	2	1	3	1	12	1	5	1	4	1	3	1	2	1	3	1	6	1	5	1	7	1
1	6	1	5	1	2	1	3	1	2	1	11	1	7	1	10	1	2	1	3	1	2	1	4
2	1	3	1	7	1	4	1	6	1	3	1	2	1	3	1	5	1	4	1	8	1	3	1
1	11	1	2	1	3	1	2	1	13	1	5	1	4	1	2	1	3	1	2	1	9	1	5
3	1	4	1	8	1	5	1	3	1	2	1	3	1	6	1	12	1	7	1	3	1	2	1

Figure 6: A 24×24 pattern [14].

Proposition 5.7. $\chi_p^{3,3}(\mathbb{Z}^2) \leq 33$.

Proof. In the pattern of Figure 6, B_i denotes the set of vertices colored by i . Note that B_2 and B_3 are both 3-packings. It can be seen that $B_{16} \cup B_{17}$ form a 11-packing and that four 7-packings form a partition of B_2 or B_3 . In order to color \mathbb{Z}^2 starting with 3, we partition B_1 into sixteen i -packings, $2 \leq i \leq 17$ (since B_1 is $\bigcup_{i=2}^{17} B_i$ translated by the vector $(1,0)$). Let B'_i denote a copy of B_i translated by $(1,0)$. We use two colors 3 to color B_2 and B_3 , and one color i in order to color B_i for $i \in [4, 8]$. We color B'_i by one color i , for $i \in [3, 8]$ and B'_2 that is a 3-packing is colored by one color 4, one color 5, one color 6 and one color 7. We use the remaining color 8 to color B_9 . We use two colors 9 in order to color B_{16} , B'_{16} , B_{17} and B'_{17} . The remaining color 9 is used to color B'_9 . We use two colors i in order to color B_i and B'_i for $i \in [10, 13]$. The remaining colors 10, 11, 12 and 13 are used to color B_{14} , B'_{14} , B_{15} and B'_{15} . \square

Proposition 5.8. $\chi_p^{3,4}(\mathbb{Z}^2) \leq 20$, $\chi_p^{3,5}(\mathbb{Z}^2) \leq 17$, $\chi_p^{3,6}(\mathbb{Z}^2) \leq 14$ and $\forall n \geq 8$, $\chi_p^{3,n}(\mathbb{Z}^2) = 8$.

Proof. Using Proposition 3.5, we define a $(3, n)$ -packing coloring of \mathbb{Z}^2 for each $n = 4, 5, 6$ and $n \geq 8$. \mathbb{Z}^2 can be partitioned into eight 3-packings, hence $\forall n \geq 8$,

$\chi_\rho^{3,n}(\mathbb{Z}^2) = 8$. The other colorings are described in Table C.11. \square

Proposition 5.9. $\chi_\rho^{4,4}(\mathbb{Z}^2) \leq 56$, $\chi_\rho^{4,5}(\mathbb{Z}^2) \leq 34$, $\chi_\rho^{4,6}(\mathbb{Z}^2) \leq 28$ and $\forall n \geq 13$, $\chi_\rho^{4,n}(\mathbb{Z}^2) = 13$.

Proof. Using Proposition 3.6, we define a $(4, n)$ -packing coloring of \mathbb{Z}^2 for each $n = 4, 5, 6$ and $n \geq 13$. \mathbb{Z}^2 can be partitioned into thirteen 4-packings, hence $\forall n \geq 13$ $\chi_\rho^{4,n}(\mathbb{Z}^2) = 13$. The other colorings are described in Table C.12. \square

5.3 Triangular lattice

Proposition 5.10. $\chi_\rho^{1,1}(\mathcal{T}) = \infty$, $\chi_\rho^{3,2}(\mathcal{T}) = \infty$, $\chi_\rho^{4,3}(\mathcal{T}) = \infty$, $\chi_\rho^{5,4}(\mathcal{T}) = \infty$, $\chi_\rho^{7,5}(\mathcal{T}) = \infty$ and $\chi_\rho^{8,6}(\mathcal{T}) = \infty$.

Proof. Let \mathcal{T} be the triangular lattice and k be an integer, $k \geq 8$.

$$\sum_{i=1}^k \frac{1}{A(i)} = \sum_{i=1}^k \frac{1}{A(2i)} + \sum_{i=0}^k \frac{1}{A(2i+1)} = \sum_{i=1}^k \frac{1}{3i^2+3i+1} + \sum_{i=0}^k \frac{1}{3i^2+6i+3} < \frac{1}{3}\sqrt{3}\pi \tanh(\frac{1}{6}\pi\sqrt{3}) + \frac{1}{18}\pi^2 - 1 < 0.854.$$

$$\text{Therefore: } \sum_{i=1}^k \frac{1}{A(i)} < 0.854 < 1, \sum_{i=3}^k \frac{2}{A(i)} < 2(0.854 - \sum_{i=1}^2 \frac{1}{A(i)}) < 0.755 < 1,$$

$$\sum_{i=4}^k \frac{3}{A(i)} < 3(0.854 - \sum_{i=1}^3 \frac{1}{A(i)}) < 0.883 < 1, \sum_{i=5}^k \frac{4}{A(i)} < 4(0.854 - \sum_{i=1}^4 \frac{1}{A(i)}) <$$

$$0.966 < 1, \sum_{i=7}^k \frac{5}{A(i)} < 5(0.854 - \sum_{i=1}^6 \frac{1}{A(i)}) < 0.887 < 1 \text{ and } \sum_{i=8}^k \frac{6}{A(i)} < 6(0.854 -$$

$$\sum_{i=1}^7 \frac{1}{A(i)}) < 0.940 < 1. \text{ Corollary 2.3 allows us to conclude. } \square$$

Proposition 5.11. $\chi_\rho^{1,2}(\mathcal{T}) \leq 6$ and $\forall n \geq 3$, $\chi_\rho^{1,n}(\mathcal{T}) = 3$.

Proof. Using Proposition 3.7, we define a $(1, n)$ -packing coloring of \mathcal{T} for each $n = 2$ and $n \geq 3$. \mathcal{T} can be partitioned into three independent sets, hence $\forall n \geq 3$, $\chi_\rho^{1,n}(\mathcal{T}) = 3$. The other coloring is described in the following table.

(1,2)-packing	P	$2 \times X_1$	X_1
	C	2×1	$4 \times A_3$
			$2 \times 2, 2 \times 3$

\square

Proposition 5.12. $\chi_\rho^{2,4}(\mathcal{T}) \leq 16$, $\chi_\rho^{2,5}(\mathcal{T}) \leq 13$, $\chi_\rho^{2,6}(\mathcal{T}) \leq 10$ and $\forall n \geq 7$, $\chi_\rho^{2,n}(\mathcal{T}) = 7$.

Proof. Using Proposition 3.8, we define a $(2, n)$ -packing coloring of \mathcal{T} for each $n = 4, 5, 6$ and $n \geq 7$. \mathcal{T} can be partitioned into seven 2-packings, hence $\forall n \geq 7$, $\chi_\rho^{2,n}(\mathcal{T}) = 7$. The other colorings are described in Table C.13. \square

Proposition 5.13. $\chi_\rho^{3,4}(\mathcal{T}) \leq 72$, $\chi_\rho^{3,5}(\mathcal{T}) \leq 38$, $\chi_\rho^{3,6}(\mathcal{T}) \leq 26$ and $\forall n \geq 12$, $\chi_\rho^{3,n}(\mathcal{T}) = 12$.

Proof. Using Proposition 3.9, we define a $(3, n)$ -packing coloring of \mathcal{T} for each $n = 4, 5, 6$ and $n \geq 12$. \mathcal{T} can be partitioned into twelve 3-packings, hence $\forall n \geq 12$, $\chi_\rho^{3,n}(\mathcal{T}) = 12$. The other colorings are described in Table C.14. \square

6 Conclusion

We have determined or bounded the (d, n) -packing chromatic number of three lattices \mathcal{H} , \mathbb{Z}^2 and \mathcal{T} for small values of d and n . Further studies can be done with other values of d and n or improving existing values. The (d, n) -packing chromatic number can also be investigated for other lattices. As an example, we can prove, using color patterns defined in [15] for distance graphs, that for the octagonal lattice \mathcal{O} , *i.e.* the strong product of two infinite path (which is a supergraph of \mathcal{T}), $\chi_\rho^{1,2}(\mathcal{O}) \leq 58$. For other finite or infinite graphs, like k -regular infinite trees, the method has to be adapted or changed since a maximized packing cannot be described as easily as those considered in this paper. Also, for each of three lattices studied, finding a sequence S such that $\chi_\rho^S = k$ and there is no S -packing k coloring where the s_1 -packing is maximized could be an interesting result.

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A Distances in the three lattices

Definition A.1 ([10]). *Let $v = (a, b)$ be a vertex in the hexagonal lattice. Then the type of v is*

$$\tau(v) = a + b + 1 \pmod{2}.$$

As $\mathcal{H} = V_0 \cup V_1$ is a bipartite graph, the type of a vertex v corresponds to the index of the set V_i to which v belongs.

Proposition A.2 ([10]). *Let $v_1 = (a_1, b_1)$, $v_2 = (a_2, b_2)$ be two vertices of the hexagonal lattice and assume that $b_1 \geq b_2$. Then the distance between v_1 and v_2 is*

$$d(v_1, v_2) = \begin{cases} |a_1 - a_2| + |b_1 - b_2| & \text{if } |a_1 - a_2| \geq |b_1 - b_2|; \\ 2|b_1 - b_2| - \tau(v_1) + \tau(v_2) & \text{if } |a_1 - a_2| < |b_1 - b_2|. \end{cases}$$

Example A.3. *The set X_2 from Figure 2 is a 2-packing in \mathcal{H} .*

Proof. Let x and y be integers, then

$$d((2(x+1)+4y, x+1), (2x+4y, x)) = |2x+4y+2-2x-4y| + |x+1-x| = 3 > 2$$

$$\text{and } d((2x+4(y+1), x), (2x+4y, x)) = 4 > 2;$$

let i and j be integers, then $d((2(x+i)+4(y+j), x+i), (2x+4y, x)) \geq \min(d((2(x+1)+4y, x+1), (2x+4y, x)), d((2x+4(y+1), x), (2x+4y, x))) = 3$, hence X_2 is a 2-packing. \square

Claim A.4. *Let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be two vertices of the square lattice. Then the distance between v_1 and v_2 is*

$$d(v_1, v_2) = |a_1 - a_2| + |b_1 - b_2|.$$

Example A.5. *The set X_2 from Figure 4 is a 2-packing in \mathbb{Z}^2 .*

Proof. Let x and y be integers, then

$d((2(x+1)+y, x+1+3y), (2x+y, x+3y)) = |2x+y+2-2x-y| + |x+1+3y-x-3y| = 3 > 2$ and $d((2x+y+1, x-1+3(y+1)), (2x+y, x+3y)) = 4 > 2$, to conclude X_2 is a 2-packing. \square

Claim A.6. Let $v_1 = (a_1, b_1)$ and $v_2 = (a_2, b_2)$ be two vertices of the triangular lattice. Then the distance between v_1 and v_2 is

$$d(v_1, v_2) = \begin{cases} \max(|a_1 - a_2|, |b_1 - b_2|) & \text{if } ((a_1 \geq a_2) \wedge (b_1 \leq b_2)) \vee ((a_1 \leq a_2) \wedge (b_1 \geq b_2)); \\ |a_1 - a_2| + |b_1 - b_2| & \text{otherwise.} \end{cases}$$

Example A.7. The set X_1 from Figure 5 is an independent set in \mathcal{T} .

Proof. Let x and y be integers, then,

$d((x+1+3y, x+1), (x+3y, x)) = |x+1+3y-x-3y| + |x+1-x| = 2 > 1$ and $d((x+3(y+1), x), (x+3y, x)) = 3 > 1$, to conclude X_1 is an independent set. \square

B Decomposition of an i -packing in the three lattices

i	Number of i -packings	Description of a i -packing	Family of translation vectors
$4k-1$	k^2	$\{3kx+6ky, kx\}$	$(3i+6j, i)$ $i, j \in \{0, \dots, k-1\}$
$6k-1$	$3k^2$	$\{3kx+6ky, 3kx\}$	$(3i+6j, 3i+2a)$ $i, j \in \{0, \dots, k-1\}, a \in \{0, 1, 2\}$
$10k-1$	$8k^2$	$\{6kx+12ky, 4kx\}$	$(6i+12j+3b, 4i+2a+b)$ $i, j \in \{0, \dots, k-1\},$ $a \in \{0, 1, 2, 3\}, b \in \{0, 1\}$
$18k-1$	$24k^2$	$\{12kx+24ky, 6kx\}$	$(12i+24j+3b, 6i+2a+b)$ $i, j \in \{0, \dots, k-1\},$ $a \in \{0, \dots, 5\}, b \in \{0, 1, 2, 3\}$

Table B.4: Decomposition of X_3 in \mathcal{H} into i -packings.

C Decomposition and associated colors

i	Number of i -packings	Description of a i -packing	Family of translation vectors
$5k - 1$	k^2	$\{3kx - ky, 2kx + 3ky\}$	$(3i - j, 2i + 3j)$ $i, j \in \{0, \dots, k - 1\}$
$6k - 1$	$2k^2$	$\{7kx - ky, kx + 3ky\}$	$(7i + 3a - j, i + 2a + 3j)$ $i, j \in \{0, \dots, k - 1\}, a \in \{0, 1\}$
$8k - 1$	$3k^2$	$\{7kx + 2ky, kx + 5ky\}$	$(7i + 2j + 3a, i + 5j + 2a)$ $i, j \in \{0, \dots, k - 1\}, a \in \{0, 1, 2\}$
$11k - 1$	$6k^2$	$\{-2kx + 11ky, 6kx\}$	$(-2i + 11j + 7a, 6i + a)$ $i, j \in \{0, \dots, k - 1\}, a \in \{0, \dots, 5\}$

Table B.5: Decomposition of X_4 in \mathcal{H} into i -packings.

	i	Number of i -packings	Description of a i -packing	Family of translation vectors
X_2	$3k - 1$	k^2	$\{2kx - ky, kx + 2ky\}$	$(2i - j, i + 2j)$ $i, j \in \{0, \dots, k - 1\}$
	$4k - 1$	$2k^2$	$\{4kx + ky, 2kx + 3ky\}$	$(4i + 2a + j, 2i + 2a + 3j)$ $i, j \in \{0, \dots, k - 1\}, a \in \{0, 1\}$
X_3	$4k - 1$	k^2	$\{2kx + 4ky, 2kx\}$	$(2i + 4j, 2i)$ $i, j \in \{0, \dots, k - 1\}$
X_4	$5k - 1$	k^2	$\{3kx - 2ky, 2kx + 3ky\}$	$(3i - 2j, 2i + 3j)$ $i, j \in \{0, \dots, k - 1\}$
	$6k - 1$	$2k^2$	$\{6kx + ky, 4kx + 5ky\}$	$(6i + j + 3a, 4i + 5j + 2a)$ $i, j \in \{0, \dots, k - 1\}, a \in \{0, 1\}$

Table B.6: Decomposition of X_2 , X_3 and X_4 in \mathbb{Z}^2 into i -packings.

	i	Number of i -packings	Description of a i -packing	Family of translation vectors
X_1	$2k - 1$	k^2	$\{kx + 3ky, kx\}$	$(i + 3j, i)$ $i, j \in \{0, \dots, k - 1\}$
	$3k - 1$	$3k^2$	$\{3kx + 3ky, 3kx\}$	$(3i + 3j + a, 3i + a)$ $i, j \in \{0, \dots, k - 1\}, a \in \{0, 1, 2\}$
X_2	$3k - 1$	k^2	$\{2kx + 7ky, kx\}$	$(2i + 7j, i)$ $i, j \in \{0, \dots, k - 1\}$
X_3	$4k - 1$	k^2	$\{2kx + 6ky, 2kx\}$	$(2i + 6j, 2i)$ $i, j \in \{0, \dots, k - 1\}$
	$6k - 1$	$3k^2$	$\{6kx + 6ky, 6kx\}$	$(6i + 6j + 2a, 6i + 2a)$ $i, j \in \{0, \dots, k - 1\}, a \in \{0, 1, 2\}$

Table B.7: Decomposition of X_1 , X_2 and X_3 in \mathcal{T} into i -packings.

(3,2)-packing	P	$2 \times X_3$	X_3 $3 \times X_5$	X_3 $4 \times X_7$	X_3 $4 \times X_9, 8 \times X_{15}$	X_3 $X_5, 3 \times X_{11},$ $4 \times X_{17}, 6 \times X_{23}$
	C	2×3	$2 \times 4, 5$	$2 \times 6, 2 \times 7$	$2 \times 8, 2 \times 9,$ $2 \times 12, 2 \times 13,$ $2 \times 14, 2 \times 15$	$5, 2 \times 10, 2 \times 11,$ $2 \times 16, 2 \times 17, 2 \times 18,$ $2 \times 19, 20$
(3,3)-packing	P	$3 \times X_3$	X_3 $3 \times X_5$	X_3 $3 \times X_5$	X_3 $4 \times X_7$	
	C	3×3	3×4	3×5	$3 \times 6, 7$	
(3,4)-packing	P	$4 \times X_3$	X_3 $3 \times X_5$	X_3 $3 \times X_5$		
	C	4×3	3×4	$4, 2 \times 5$		
(3,5)-packing	P	$5 \times X_3$	X_3 $3 \times X_5$			
	C	5×3	3×4			

Table C.8: Decomposition of \mathcal{H} into 3-packings and associated colors.

(4,3)-packing	P	$3 \times X_4$	X_4 $2 \times A_5$	$2 \times X_4$ $6 \times A_7$	$2 \times X_4$ $A_5, 6 \times A_9$	X_4 $6 \times A_{11}, 4 \times A_{19}$
	C	3×4	2×5	$3 \times 6, 3 \times 7$	$5, 3 \times 8, 3 \times 9$	$3 \times 10, 3 \times 11, 18, 3 \times 19$
	P				X_4 $9 \times A_{14}$	X_4 $11 \times A_{19}, 10 \times A_{23}$
	C				$3 \times 12, 3 \times 13,$ 3×14	$3 \times 15, 3 \times 16, 3 \times 17,$ $2 \times 18, 3 \times 20, 3 \times 21$ $3 \times 22, 23$
(4,4)-packing	P	$4 \times X_4$	$2 \times X_4$ $4 \times A_5$	$2 \times X_4$ $6 \times A_7$	$2 \times X_4$ $8 \times A_9$	X_4 $2 \times A_7, 3 \times A_{14}$
	C	4×4	4×5	$4 \times 6, 2 \times 7$	$4 \times 8, 4 \times 9$	$2 \times 7, 3 \times 10$
(4,5)-packing	P	$5 \times X_4$	$2 \times X_4$ $4 \times A_5$	$3 \times X_4$ $9 \times A_7$	X_4 $A_5, 2 \times A_9$	
	C	5×4	4×5	$5 \times 6, 4 \times 7$	$5, 7, 8$	
(4,6)-packing	P	$6 \times X_4$	$3 \times X_4$ $6 \times A_5$	$2 \times X_4$ $6 \times A_7$		
	C	6×4	6×5	6×6		

Table C.9: Decomposition of \mathcal{H} into 4-packings and associated colors.

(2,2)-packing	P	$2 \times X_2$	X_2 $2 \times A_3$	X_2 $4 \times A_5$	X_2 $6 \times A_8, 6 \times A_{11}$
	C	2×2	2×3	$2 \times 4, 2 \times 5$	$2 \times 6, 2 \times 7, 2 \times 8$ $2 \times 9, 2 \times 10, 2 \times 11$
(2,3)-packing	P	$3 \times X_2$	X_2 $2 \times A_3$	X_2 $A_3, 2 \times A_5$	
	C	3×2	2×3	$3, 2 \times 4$	
(2,4)-packing	P	$4 \times X_2$	X_2 $2 \times A_3$		
	C	4×2	2×3		

Table C.10: Decomposition of \mathbb{Z}^2 into 2-packings and associated colors.

(3,4)-packing	P	$4 \times X_3$	$4 \times X_3$ $16 \times A_7$
	C	4×3	$4 \times 4, 4 \times 5, 4 \times 6, 4 \times 7$
(3,5)-packing	P	$5 \times X_3$	$3 \times X_3$ $12 \times A_7$
	C	5×3	$5 \times 4, 5 \times 5, 2 \times 6$
(3,6)-packing	P	$6 \times X_3$	$2 \times X_3$ $8 \times A_7$
	C	6×3	$6 \times 4, 2 \times 5$

Table C.11: Decomposition of \mathbb{Z}^2 into 3-packings and associated colors.

(4,4)-packing	P	$4 \times X_4$	$2 \times X_4$ $4 \times A_5$	$4 \times X_4$ $16 \times A_9$	X_4 $8 \times A_{11}$	X_4 $9 \times A_{14}$	X_4 $3 \times A_{14}, 12 \times A_{17}$
	C	4×4	4×5	$4 \times 6, 4 \times 7,$ $4 \times 8, 4 \times 9$	$4 \times 10,$ 4×11	4×12 $4 \times 13, 14$	$3 \times 14, 4 \times 15,$ $4 \times 16, 4 \times 17$
(4,5)-packing	P	$5 \times X_4$	$2 \times X_4$ $4 \times A_5$	$5 \times X_4$ $A_5, 18 \times A_9$	X_4 $2 \times A_9,$ $4 \times A_{11}$		
	C	5×4	4×5	$5, 5 \times 6, 5 \times 7$ $5 \times 8, 3 \times 9$	$2 \times 9,$ 4×10		
(4,6)-packing	P	$6 \times X_4$	$3 \times X_4$ $6 \times A_5$	$4 \times X_4$ $16 \times A_9$			
	C	6×4	6×5	$6 \times 6, 6 \times 7, 4 \times 8$			

Table C.12: Decomposition of \mathbb{Z}^2 into 4-packings and associated colors.

(2,4)-packing	P	$4 \times X_2$	$3 \times X_2$
	C	4×2	$12 \times A_5$
(2,5)-packing	P	$5 \times X_2$	$2 \times X_2$
	C	5×2	$8 \times A_5$
(2,6)-packing	P	$6 \times X_2$	X_2
	C	6×2	$4 \times A_5$

Table C.13: Decomposition of \mathcal{T} into 2-packings and associated colors.

(3,4)-packing	P	$4 \times X_3$	$2 \times X_3$	$2 \times X_3$	$2 \times X_3$
	C	4×3	$6 \times A_5$	$8 \times A_7$	$2 \times A_5, 12 \times A_{11}$
	P			X_3	X_3
	C			$16 \times A_{15}$	$4 \times A_{11}, 20 \times A_{23}$
(3,5)-packing	P	$5 \times X_3$	$3 \times X_3$	$2 \times X_3$	$2 \times X_3$
	C	5×3	$9 \times A_5$	$8 \times A_7$	$18 \times A_{11}$
(3,6)-packing	P	$6 \times X_3$	$4 \times X_3$	$2 \times X_3$	
	C	6×3	$12 \times A_5$	$8 \times A_7$	

Table C.14: Decomposition of \mathcal{T} into 3-packings and associated colors.